Condensate turbulence in two dimensions

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The nonlinear Schrödinger equation with repulsion (also called the Gross-Pitaevsky equation) is solved numerically with damping at small scales and pumping at intermediate scales and without any large-scale damping. Inverse cascade creating a wave condensate is studied. At moderate pumping, it is shown that the evolution comprises three stages: (i) short period (few nonlinear times) of setting the distribution of fluctuations with the flux of waves towards large scales, (ii) long intermediate period of self-saturated condensation with the rate of condensate growth being inversely proportional to the condensate amplitude, the number of waves growing as $\sqrt{t}$, the total energy linearly increasing with time and the level of over-condensate fluctuations going down as $1/\sqrt{t}$, and (iii) final stage with a constant level of over-condensate fluctuations and with the condensate linearly growing with time. Most of the waves are in the condensate. The flatness initially increases and then goes down as the over-condensate fluctuations are suppressed. At the final stage, the second structure function $\langle|\psi_1-\psi_2|^2\rangle\propto \ln r_{12}$ while the fourth and sixth functions are close to their Gaussian values. Spontaneous symmetry breaking is observed: turbulence is much more anisotropic at large scales than at pumping scales. Another scenario may take place for a very strong pumping: the condensate contains 25–30 % of the total number of waves, the harmonics with small wave numbers grow as well. [S1063-651X(96)06911-5]

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If an unforced undamped system conserves more than one integral of motion then the pumping acting at some scale generally produces two cascades in k space. While a downscale (direct) cascade and related fragmentation is what one naturally expects from turbulence, an upscale (inverse) cascade is a kind of self-organization. Another important distinction between cascades stems from their destinations. Most of the systems provide for a small-scale dissipation (like viscosity) as a natural sink for a direct cascade. Contrary, in many systems, large-scale modes do not have significant damping so that inverse cascade may lead to a growth of the largest mode.

Besides the inverse cascade in two-dimensional incompressible fluid [1–3], any system with the Hamiltonian $H=H_2+H_4$,

$$H_2=\sum_k \omega_k\psi_k^*\psi_k\,,\quad H_4=\sum_{k_1k_2k_3k_4}\lambda_{k_1k_2k_3k_4}\psi_{k_1}^*\psi_{k_2}^*\psi_{k_3}\psi_{k_4},$$

allows for an inverse cascade since it conserves the total number of waves $N=\Sigma_k|\psi_k|^2$. The system is a collection of waves with the frequency spectrum ω_k and the four wave interaction with the evident properties $\lambda_{k_1k_2k_3k_4}=\lambda_{k_2k_1k_3k_4}=\lambda_{k_3k_4k_1k_2}$. At a small level of nonlinearity $H_{int}\ll H_2$, the weak turbulence of such a system is well understood [4]. The presence of two integrals of motion, both quadratic in wave amplitudes (neglecting the contribution of H_{int} into the Hamiltonian), leads to the existence of two cascades, direct energy cascade and inverse cascade of waves, leading to a large-scale build-up that will be called the condensate. The simple theorem can be readily proved: one needs at least two sinks (one at larger and another at smaller scales than that of the pumping) to absorb both integrals and provide for a steady state [4]. At strong nonlinearity, the relations between integrals and their fluxes are not so simple, as we shall see below.

We consider the case of $\omega_k=k^2$ and $\lambda_{k_1k_2k_3k_4}=\text{const}$ corresponding to the famous nonlinear Schrödinger equation,

$$i\psi_t+\Delta\psi+\lambda|\psi|^2\psi=0, \tag{1}$$

which describes light in media with Kerr nonlinearity as well as any turbulence of envelopes [5]. Weak turbulence in the framework of (1) is described in [4,6–8]. In two dimensions, the main prediction for the pair correlation function $\langle\psi_k\psi_{k'}^*\rangle=n_k\Delta_{\mathbf{kk'}}$ is as follows: $n_k=f(k)T/k^2$, where the dimensionless function $f(k)$ (which distorts equilibrium distribution T/k^2 to provide for nonzero fluxes) is a slow logarithmic function at the region of the direct cascade [8] and it approaches a constant at the region of the inverse cascade [7]. The dimensionless parameter of nonlinearity $\lambda|a_k|^2k^2/\omega_k=\lambda|a_k|^2$ increases with the wavelength so that the turbulence is getting strong at small k. Contrary to weak turbulence, which is insensitive to the sign of the interaction constant λ, strong turbulence has qualitatively different properties for focusing ($\lambda>0$) and defocusing media ($\lambda<0$).

Turbulence at the focusing case has been qualitatively described in [7]: an inverse cascade produces large-scale caverns that are modulationally unstable and collapse in a finite time. Collapse (or self-focussing) provides for a strongly nonlinear direct cascade of waves towards the small-scale sink. Eventually, a steady state is established without an external large-scale sink: all mean values (energy, total number

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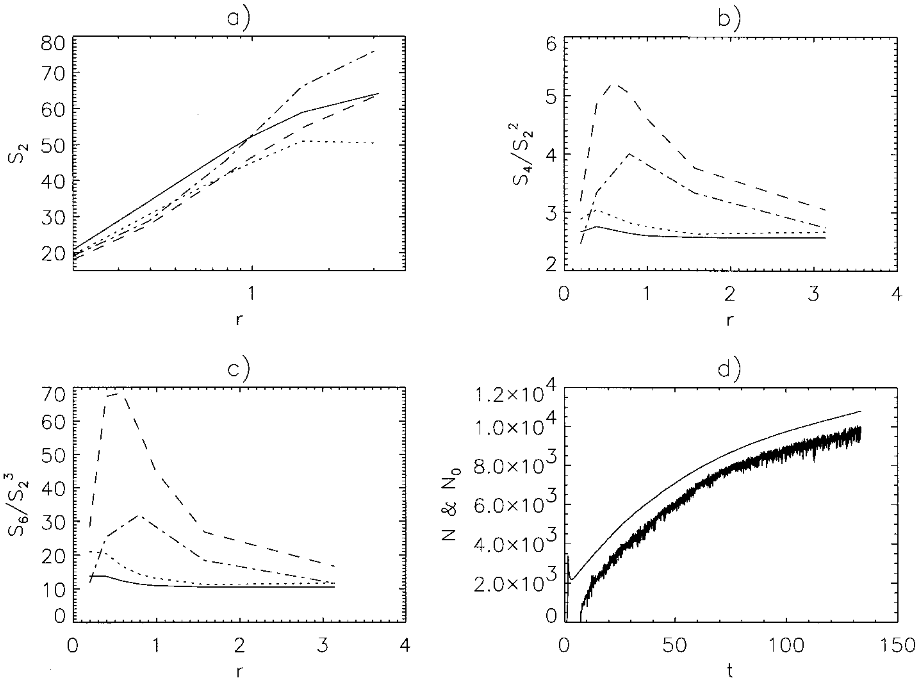

FIG. 1. (a)–(c) Structure functions at different times: $-\cdot-t=32$, $--t=55$, $\cdots t=92$, $—t=127$; (d) total number of waves N and the number of waves in the coherent condensate N_0.

of waves, etc.) do not grow with time. Strong large-scale turbulence coexists with small-scale weak turbulence.

We consider the defocusing case. Equation (1) with $\lambda<0$ is called the Gross-Pitaevsky equation [9]. Immense literature is devoted to the properties of particular solutions and to equilibrium statistics (including phase transitions) in that model; see, e.g., [10–12]. We are interested in turbulence excited by an external pumping. Strong turbulence in the framework of (1) may contain a complicated mixture of condensate, phonons, shocks, grey solitons, and quantized vortices. Such complexity does not imply the absence of simple universal scaling laws for the correlation functions, yet it makes it difficult to establish them. The complete analytical description of such turbulence is still ahead of us. The three-dimensional kinetics has been considered by Kraichnan within the ring-model approximation [13]. We consider two-dimensional turbulence under the action of the instability providing pumping at intermediate scales and the small-scale damping. This paper is an account of the first step, we adress the simplest questions: Is it possible to have a steady state at finite k without a large-scale sink? When and how does the condensate appear? We shall show that those two questions are closely connected: fluctuations at $k\neq 0$ can be steady only if a growing condensate appears at $k=0$.

For the numerical integration of Eq. (1) with $\lambda=-1$, we applied the method described in [7]. All runs were done in the domain $2\pi\times 2\pi$. For the initial conditions, the ‘‘quasiequlibrium’’ spectrum $n_k=T/(k^2+\mu)$ was chosen with $T=0.01$ and $\mu=1/12$. The initial phases were taken randomly. To consider turbulence, we added to the right-hand of the equation for $\partial\psi_k/\partial t$ the terms $\gamma_p(k)\psi_k-\gamma_d(k)\psi_k$ which describe the pumping due to instability,

$$\gamma_p(k)=\alpha\sqrt{(k^2-k_l^2)(k_r^2-k^2)} \quad \text{at} \quad k_l<k<k_r$$

(and zero otherwise) and the small-scale damping

$$\gamma_d(k)=0.5k^2h(k/k_d),$$

$$h(x)=\frac{1}{6x^5}\exp[5(1-x^{-2})], \quad x\leqslant 1,$$

$$h(x)=1-\frac{5}{6}\exp[1/2(1-x^2)], \quad x>1.$$

The choice of γ_d was to model Landau damping in plasma; the particular form of $h(x)$ should be irrelevant as long as $h(x)$ tends to a constant at large x.

Figures 1 and 2 present the data for the run on a 128×128 grid with $\alpha=0.05, k_l=28, k_r=32$ (moderate pumping). One can see from Fig. 1(d) that the condensate appears after $t\simeq 10$ and eventually the most of the waves are in the condensate. Here N_0 has been calculated from the average value of ψ over space, i.e., it represents a coherent condensate. The square-root regime $N_0\propto\sqrt{t}$ starts when the number of waves at the condensate is of order of the total number of waves. The regime ends (at $t\approx 70$) when the correlation scale $r_0\simeq N_0^{-1/2}$ is approaching the dissipation scale. To characterize the overcondensate fluctuations, the structure functions $S_i(r)=\langle|\psi(\mathbf{x}+\mathbf{r})-\psi(\mathbf{x})|^i\rangle$ were obtained by averaging over $\mathbf{x}$ and over time (during an interval of few dimen-

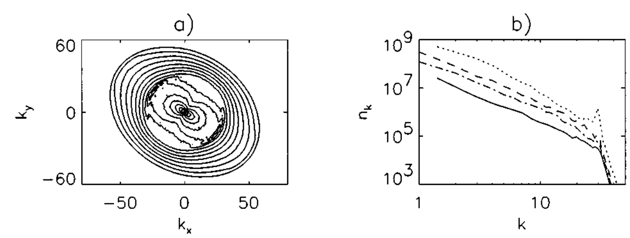

FIG. 2. (a) Level sets of the spectrum $n(k_x,k_y)$ averaged over time $t=(119–127)$. (b) Spectra at different directions in k space: $-\cdot-\ k_x=0$, $---\ k_y=0$, $\cdots\ k_x=-k_y$, unbroken line $k_x=k_y$.

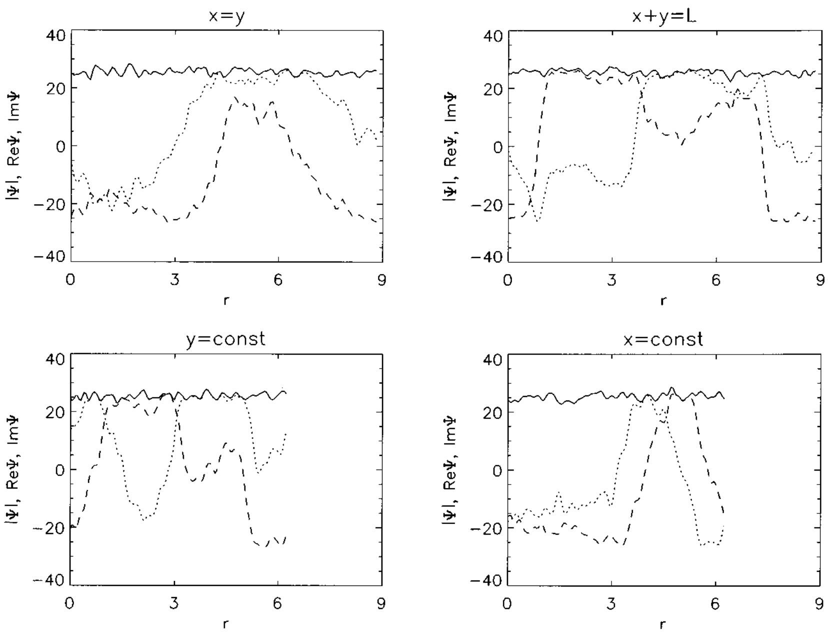

FIG. 3. Profiles of $|\Psi|$ (solid line), ReΨ (broken line), and ImΨ (dotted line) along the diagonals and parallel to the box walls, $t=135.7$

sionless time units). One can see from Fig. 1(a) that the second structure function is getting logarithmic at large time when it stabilizes (after the time $t\simeq100$ the variations of S_i are invisible on the scale of Fig. 1). That corresponds to the spectrum $n_k\propto k^{-2}$. To see if the statistics of the overcondensate fluctuations deviates from Gaussian statistics we calculated the second flatness S_4/S_2^2 (2 for Gaussian statistics) and the third one S_6/S_2^3 (8 for Gaussian statistics). The flatness grows up until the time $t\approx55$ (when the relative growth of the condensate saturates), then the flatness decreases. The flatness is approximately scale-independent at $r>r_0$ where the correlation scale $r_0\simeq N_0^{-1/2}$ decreases with time.

Steady spectrum is substantially anisotropic despite the pumping being almost isotropic; the anisotropy grows as k decreases — see Fig. 2(a). Note that within the theory of weak turbulence the large-scale equilibrium spectrum is structurally stable with respect to the angular modulations [4]. That means that the symmetry breaking may be due to strongly nonlinear phenomena (like kink creation) — see also Fig. 3. Figures 2 and 3 show that there are two perpendicular directions, one (at $\simeq30°$) with the largest level of fluctuations and another with the smallest level. At the first direction, the spectrum is flat at small k and then drops as approximately k^{-3} which may be the spectrum of shocks. The small fluctuations at the perpendicular direction have spectrum k^{-2}. One may hypothetize that the strong turbulence of overcondensate fluctuations may be considered as quasi-one-dimensional structures imposed onto a more or less isotropic set of waves.

To study the asymptotic regime (with linearly growing condensate and a steady level of fluctuations) on a larger

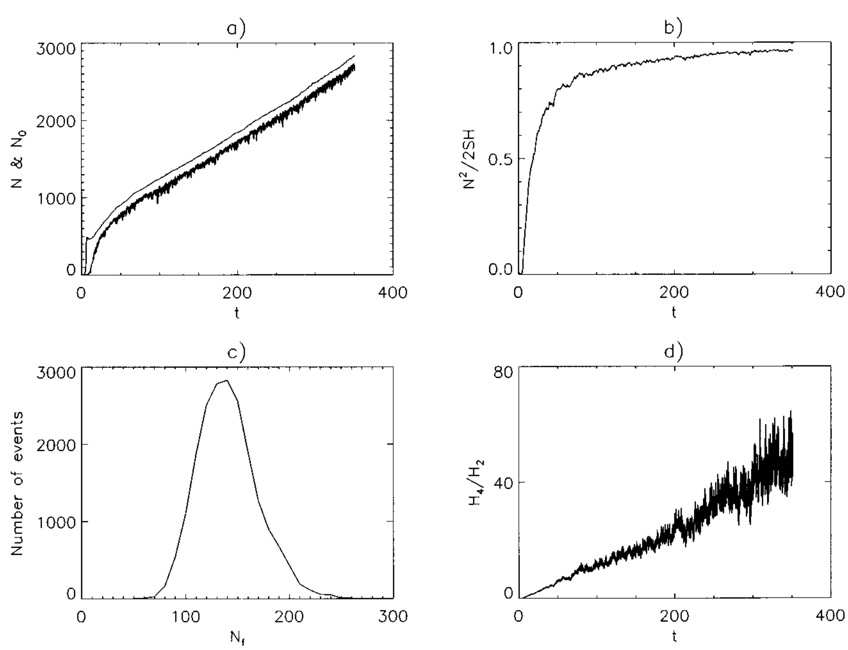

FIG. 4. Long-time evolution under the action of a moderate pumping: (a) total number of waves N and the number of waves in the coherent condensate N_0, (b) ratio of number of waves squared to the Hamiltonian, (c) hystogram of the overcondensate fluctuations, (d) nonlinearity parameter $H_4/H_2=\int|\psi|^4dxdy/2\int|\nabla\psi|^2dxdy$.

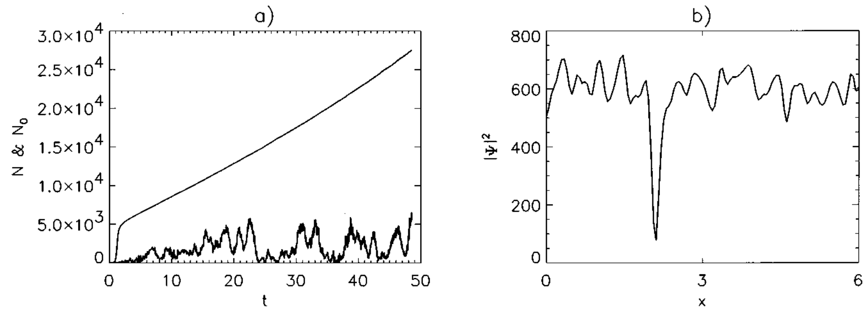

FIG. 5. (a) Initial stage of the evolution under the action of a strong pumping: total number of waves N and the number of waves in the coherent condensate N_0, (b) spatial dependence of $|\psi(x,y)|^2$ at $t=42.4$.

time scale, we made a run on 32×32 grid with $\alpha=0.05, k_l=6, k_r=10$ (see Fig. 4). Note that the parameter of nonlinearity H_4/H_2 is much larger than unity when the fluctuations are steady, yet the ratio $N^2/2SH$ ($S=4\pi^2$ being the domain area) is close to unity so that the condensate gives the main contribution into both energy and number of waves. Figure 4(c) presents the hystogram of the fluctuations of the number of waves above the condensate. The statistics of the over-condensate fluctuations approaches Gaussian similarly to what is shown at Figs. 1(b) and 1(c), despite the high nonlinearity of the system.

Making another run (not shown) with the only difference being that the amplitude of the pumping is ten times larger, we did not find any sign of a growing condensate and of a steady state at finite k at the same time scale $t\simeq 400$. One may suggest that if the pumping initially produces weak turbulence at $k\sim k_p$ then the condensate has time to appear, and then the growing condensate serves as a sink providing for a steady state at $k=0$. Very strong pumping produces unsteady turbulence with a small condensate (at least as an intermediate regime for a long time).

Another setting where the evolution is qualitatively different from what is shown at Figs. 1–4 is when the pumping is sufficiently strong and separated from the damping region. Figures 5 and 6 show the data for the run with $\alpha=0.068, k_l=13, k_r=17$. The main difference is the position of the pumping maximum which is now $k_p=15$. The initial data and the damping are the same as in Fig. 1. As well as in the case of a strong pumping, we do not observe a steady state at nonzero k (we run until $t\simeq 70$). The total number of waves increases exponentially until $t\approx 2$, then the intermediate stage follows with an approximately quadratic growth $N\sim t^2$. After $t\simeq 50$, the evolution of $N(t)$ comes eventually into linear growth $N\sim t$. The occupation numbers n_k grow at small k and saturate at the pumping scale, the spectrum is getting steeper at $k\leqslant k_p$ with the scaling exponent approximately 4.5 at $t=44$. During the intermediate stage, the condensate contains a negligible portion of the total number of waves [Fig. 5(a)]. Only at $t>50$, the condensate starts to grow, the ratio N_0/N fluctuates around 1/3 (not shown). That means that ψ is small yet the value of $|\psi(\mathbf{r})|^2$ is almost constant in space [see Figs. 5(b) and 6]. There are only few deep minima in $|\psi|^2(\mathbf{r})$ [like that seen at Fig. 5(b) at $x\approx 2.1$] which probably correspond to grey solitons or vortices. Most of the realizations look like Fig. 6 (compare with Fig. 3 for a moderate pumping). One may see kinklike structures in the profiles of ReΨ and Imψ while $|\psi|$ is almost constant. The fact that the fluctuations of $|\psi(\mathbf{r})|^2$ in space are small is reflected also in the dependence of $N^2/2SH$ on time (not shown) which approaches unity similarly to Fig. 3(b).

To conclude, there seem to be two qualitatively different regimes of the kinetics of the nonequilibrium condensation.

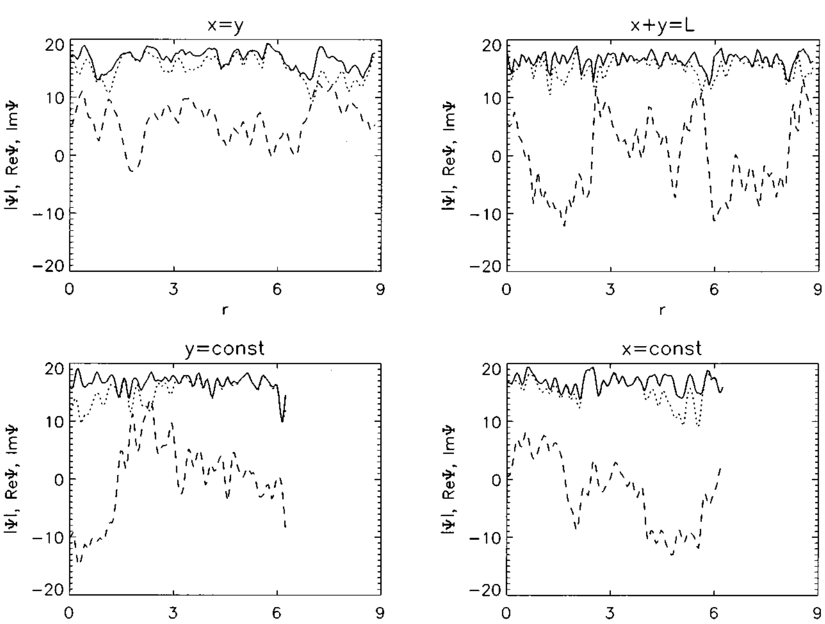

FIG. 6. Profiles of $|\Psi|$ (solid line), ReΨ (broken line), and ImΨ (dotted line) along the diagonals and parallel to the box walls, $t=45.4$.

For a moderate pumping that creates weaklike turbulence at pumping scales, the inverse cascade produces strong condensate that suppresses the over-condensate fluctuation. Linearly growing condensate provides a sink for a turbulence which is steady at any $k \neq 0$. The regimes that appear at strong pumping, where finite Fourier harmonics grow as well as the condensate, need further studies.

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